A Polynomial Time Algorithm for Rank-1 Bimatrix Games

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Two player normal form game is the most basic form of game, studied extensively in game theory. Such a game can be represented by two payoff matrices (A, B), one for each player, hence they are also known as bimatrix games. The rank of a bimatrix game (A, B) is defined as the rank of matrix $(A+B)$. For zero-sum games, i.e., rank-0, von Neumann (1928) [13] showed that Nash equilibrium are min-max strategies, which is equivalent to the linear programming duality [5, 1]. Kannan and Theobald (2005) [9, 10] gave an FPTAS for constant rank games and asked if there exists a polynomial time algorithm even for rank-1 games. The main difficulty is that unlike rank-0 games, rank-1 games can have disconnected set of equilibria; even exponentially many [15]. Adsul, Garg, Mehta and Sohoni (2011) [2] settled this question together with other interesting structural results. In doing so they extend the polynomial time solvability of linear programming to a larger class of optimization problems with non-convex, even disconnected solution set. To the best of our knowledge no such result is known when the solution set is disconnected.

This note is aimed at giving a simpler exposition of the AGMS [2] algorithm and analysis. We note that Nash equilibrium computation in general bimatrix game is PPAD-complete [12, 4, 6].

1 Preliminaries

A bimatrix game is a single shot, two player game, each player having finitely many strategies (moves) to play from. Let S_1 and S_2 denote the set of pure strategies for the first and the second player respectively, and let $m = |S_1|$ and $n = |S_2|$. Such a game can be represented by two payoff matrices A and B such that, if the played strategy profile is $(i, j) \in S_1 \times S_2$, then the payoffs of the first player is A_{ij} and that of second player is B_{ij}. Note that the rows of these matrices correspond to the strategies of the first player and the columns to the strategies of second player.

Players may randomize among their strategies; a randomized play is called a *mixed strategy.* The set of mixed strategies for the first-player is $X = \{(x_1, \ldots, x_m) \mid x_i \geq 0,\ \forall i \in S_1,\ \sum_{i \in S_1} x_i = 1\}$ and for the second-player, it is $Y = \{(y_1, \ldots, y_n) \mid y_j \geq 0,\ \forall j \in S_2,\ \sum_{j \in S_2} y_j = 1\}$. By playing $(x, y) \in X \times Y$ we mean strategies are picked independently at random as per x by the first-player and as per y by the second-player. Therefore the expected payoffs of the first-player and second-player are, respectively

$$\sum_{i,j} A_{ij} x_i y_j = x^T A y \qquad \text{and} \qquad \sum_{i,j} B_{ij} x_i y_j = x^T B y$$

Definition 1 *(Nash Equilibrium [14]) A strategy profile is said to be a Nash equilibrium strategy profile (NESP) if no player achieves a better payoff by a unilateral deviation [11]. Formally,* $(x, y) \in X \times Y$ *is a NESP iff* $\forall x' \in X,\ x^T A y \geq x'^T A y$ *and* $\forall y' \in Y,\ x^T B y \geq x^T B y'$.

Given strategy y for the second-player, the first-player gets $(Ay)_k$ from her k^{th} strategy. Clearly, her best strategies are $\arg\max_k (Ay)_k$, and a mixed strategy fetches the maximum payoff only if she randomize among her best strategies. Similarly, given x for the first-player, the second-player gets $(x^T B)_k$ from k^{th} strategy, and same conclusion applies. These can be equivalently stated as the following complementarity type conditions,

$$\begin{array}{lll} \forall i \in S_1,\ x_i > 0 & \Rightarrow & A_i y = \max_{k \in S_1} A_k y \\ \forall j \in S_2,\ y_j > 0 & \Rightarrow & x^T B^j = \max_{k \in S_2} x^T B^k \end{array} \tag{1}$$

The strategies fetching maximum payoff are called the *best response* strategies (w.r.t. the opponent's play). Based on conditions (1) next we define *best response polyhedra* (BRPs), Q for the first-player and P for the second-player. In the following expression, x and y are vector variables representing the mixed-strategies, and π_1 and π_2 are scalar variables supposed to capture payoffs.

$$\begin{array}{lll} P = \{(x, \pi_2) \in \mathbb{R}^{m+1} & | & x \in X; \quad (x^T B)_j - \pi_2 \leq 0, \quad \forall j \in S_2\} \\ Q = \{(y, \pi_1) \in \mathbb{R}^{n+1} & | & y \in Y; \quad (Ay)_i - \pi_1 \leq 0, \quad \forall i \in S_1\} \end{array} \tag{2}$$

Consider the following complementarity conditions for a point $((x, \pi_2), (y, \pi_1)) \in P \times Q$.

$$x_i((Ay)_i - \pi_1) = 0,\ \forall i \in S_1 \quad \text{and} \quad y_j((x^T B)_j - \pi_2) = 0,\ \forall j \in S_2 \tag{3}$$

For any given point in $P \times Q$, clearly, we have $x_i((Ay)_i - \pi_1) \leq 0, \forall i$ and $y_j((x^T B)_j - \pi_2) \leq 0, \forall j$. Summing all of them up gives $x^T(A + B)y - \pi_1 - \pi_2 \leq 0$. Clearly, it is exactly zero if and only if (3) is satisfied. Therefore, the following quadratic program captures points satisfying (3).

$$\begin{array}{lll} \max & : & x^T(A + B)y - \pi_1 - \pi_2 \\ s.t. & & (x, \pi_2) \in P; \quad (y, \pi_1) \in Q \end{array} \tag{4}$$

The next lemma follows from the above discussion and the fact that every NESP satisfies (1).

Lemma 2 *The following are equivalent*

1. *(x, y) is a NESP of game (A, B) with payoffs $\pi_1 = x^T Ay$ and $\pi_2 = x^T By$.*
2. *$(x, \pi_2) \in P$, $(y, \pi_1) \in Q$, and they satisfy (3), equivalently $x^T(A + B)y - \pi_1 - \pi_2 = 0$.*
3. *(x, y, π_1, π_2) is an optimal solution of the quadratic program (4).*

2 Rank-1 Game Space

Rank of a bimatrix game (A, B) is $rank(A + B)$. Zero-sum games have rank-0, and are polynomial time solvable, as they are equivalent to linear programs [5, 1]. In this section we consider the smallest extension of zero-sum games, namely rank-1 games, and analyze them with a perspective of designing a polynomial time algorithm to solve them[1].

If the game has rank-1, then matrix $A + B$ can be represented as $a \cdot b^T$, where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$. Note that tuple (A, a, b) is enough to represent such a game, since B can be written as $-A + a \cdot b^T$. Replacing $A + B$ by $a \cdot b^T$ and B by $-A + a \cdot b^T$ in (4), we get,

$$\begin{array}{lll} \max & : & (x^T \cdot a)(b^T \cdot y) - \pi_1 - \pi_2 \\ s.t. & & (y, \pi_1) \in Q \\ & & x \in X; \quad \forall j \in S_2,\ -(x^T A)_j + (x^T \cdot a) b_j \leq \pi_2 \end{array} \tag{5}$$

The formulation of (5) is a rank-1 quadratic program. Solving a rank-1 quadratic programs is NP-hard in general [7]. One major issue with rank-1 games is that they may have disconnected set

[1] By solving a game, we mean computing its Nash equilibrium.

of equilibria. We consider space S of rank-1 games, defined next, so that all of their equilibria form a well-behaved set.

$$S = \{(A, u, b) \mid u \in \mathbb{R}^m\}$$

The quadratic program (QP) of (5) is the same for all the games $(A, u, b) \in S$ except for term $(x^T \cdot a)$ appearing in the cost function and in inequality of polyhedron Q. This term becomes $(x^T \cdot u)$ and changes as u changes. Let us replace this term by a scalar variable λ at both the places,

$$\begin{array}{lll} \max & : & \lambda(b^T \cdot y) - \pi_1 - \pi_2 \\ s.t. & & (y, \pi_1) \in Q; \quad (x, \pi_2, \lambda) \in P' \end{array} \tag{6}$$

$$\text{where } P' = \{(x, \pi_2, \lambda) \mid x \in X; \ \ -(x^T A)_j + \lambda d_j \leq \pi_2, \forall j \in S_2\} \tag{7}$$

Similar to (3) consider the following complementarity conditions for $P' \times Q$,

$$x_i((Ay)_i - \pi_1) = 0, \ \forall i \in S_1 \quad \text{and} \quad y_j(-(x^T A)_j + \lambda b_j - \pi_2) = 0, \ \forall j \in S_2 \tag{8}$$

Using Lemma 2 and the above construction we show the next lemma.

Lemma 3 *The following are equivalent*

1. *(x, y) is a NESP of game $(A, u, b) \in S$ with payoffs $\pi_1 = x^T Ay$ and $\pi_2 = x^T(-A + u \cdot b^T)y$.*
2. *For $\lambda = x^T \cdot u$, we have $(x, \pi_2, \lambda) \in P'$, $(y, \pi_1) \in Q$, and they satisfy (8), equivalently $\lambda(d^T \cdot y) - \pi_1 - \pi_2 = 0$.*
3. *For $\lambda = x^T \cdot u$, $(x, y, \pi_1, \pi_2, \lambda)$ is an optimal solution of QP (6).*

Proof : $(1 \Rightarrow 2)$ Lemma 2 implies that (x, y) satisfies (3) and $(x, \pi_2) \in P$. Replacing B with $-A + u \cdot b^T$ the second matrix of game (A, u, b) and then $x^T \cdot u$ with λ in (3) gives (8), and in P of (1) gives P' of (7). Summing all of the equalities of (8) gives $\lambda(b^T \cdot y) - \pi_1 - \pi_2 = 0$.

$(2 \Rightarrow 3)$ At any feasible point of the QP (6) we have $x_i((Ay)_i - \pi_1) \leq 0, \ \forall i$ and $y_j(-(x^T A)_j + \lambda b_j - \pi_2) \leq 0, \forall j$. Summing all of these up gives $\lambda(b^T \cdot y) - \pi_1 - \pi_2 \leq 0$, and equality holds if (8) is satisfied. Further, from the first part we know that there are points in $P' \times Q$ where (8) is satisfied.

$(3 \Rightarrow 1)$ Let $v = (x, y, \pi_1, \pi_2, \lambda)$ be an optimal solution on (6); v is feasible point with objective value zero. Since, replacing λ with $x^T \cdot u$ gives the (4) for the game $(A, -A + u \cdot b^T)$, point $((y, \pi_1), (x, \pi_2))$ is feasible in it and with objective value zero. Hence proved using Lemma 2. □

Let $\mathcal{N}$ denote the set of solutions of QP (6). Except for term $\lambda(b^T \cdot y)$ everything else is linear in this QP. If we replace λ by a number, say $c \in \mathbb{R}$, then it indeed becomes a linear program (LP). Therefore, let $LP(\lambda)$ be the linear program obtained by replacing λ with the given parameter at both the places in (6).

Lemma 4 *$(x', y', \pi_1', \pi_2', \lambda') \in \mathcal{N}$ iff (x', y', π_1', π_2') is an optimal solution of $LP(\lambda')$.*

Proof : The polyhedron of $LP(c)$ for any $c \in \mathbb{R}$, is polyhedron of the QP (6) intersected with hyper-plane $\lambda = c$; a subset of $P' \times Q$ where λ takes value c. Therefore, the optimal value of $LP(c)$ is at most zero (Lemma 3).

$(\Rightarrow)$ At (x', y', π_1', π_2') the cost function of $LP(\lambda')$ evaluates to zero, and has to be optimal.

$(\Leftarrow)$ Let (x', y') be a Nash equilibrium of rank-1 game (A, u, b) where $u = [\lambda', \dots, \lambda']^T$. Clearly, $x^T \cdot u$ is λ' for all $x \in X$. Therefore, using Lemma 3 we have that $\lambda'(b^T \cdot y') - \pi_1' - \pi_2' = 0$ and (x', y', π_1', π_2') is feasible in $LP(\lambda')$, hence also optimal. The lemma follows since cost function evaluates to zero at every solution of $LP(\lambda')$ and its feasible set padded with λ' is a subset of $P' \times Q$. □

3 A Polynomial-Time Algorithm

Given a rank-1 game (A, a, b), let I denote the interval $[\min_i a_i,\ \max_i a_i]$. Since, $x \in X$ is a probability distribution vector, we have $x^T \cdot c \in I$. Consider function $f : I \rightarrow I$ such that

$$f(c) = x^T \cdot a, \quad \text{where } (x, y, \pi_1, \pi_2) \text{ is a solution of LP(c)} \tag{9}$$

If there are multiple values of x in the solution set of $LP(c)$ then it is not clear which x to use to compute $f(c)$. To avoid any such ambiguity we need to show that x takes a unique value in all the solutions of $LP(c)$. For this we will use the structure of $\mathcal{N}$ and convexity of the solution set of $LP(c)$. Let us assume that polyhedra P' and Q are non-degenerate.

Lemma 5 *Given an $a \in \mathbb{R}$, there is a unique x in the solutions of $LP(c)$.*

Proof : Note that the polyhedron $P' \times Q$ is in $m+n+1$ dimension; two equalities compensate for two variables. At a point $v \in \mathcal{N}$, either $x_i = 0$ or $(Ay)_i - \pi_1 = 0$ for every $i \in S_1$, and either $y_j = 0$ or $-(x^T A)_j + \lambda b_j - \pi_2 = 0$ for every $j \in S_2$ (Lemma 3). In other words at least $m+n$ inequalities of $P' \times Q$ hold with equality at v. Therefore, v has to be either on an edge or at a vertex of polyhedron $P' \times Q$. If it is a vertex, then exactly $m+n+1$ inequalities are tight at v, implying that for some i or j both the corresponding inequalities are tight. If it is an i then the edges we obtain by relaxing $x_i = 0$ or $(Ay)_i - \pi_1 = 0$ still satisfy (8) and hence are in $\mathcal{N}$. Similarly for j. Putting all of these together we get that $\mathcal{N} \subset$ 1-skeleton of $P' \times Q$, every vertex of which has degree two in $\mathcal{N}$. Essentially set $\mathcal{N}$ forms paths and cycles on 1-skeleton of $P' \times Q$, with unbounded edges at either end of every path.

Let $\mathcal{N}(a)$ denote the set of points of $\mathcal{N}$ with $\lambda = a$. This set is exactly the solutions of $LP(a)$ (Lemma 4), and hence is convex. In that case, $\mathcal{N}$ can have only paths, where λ changes monotonically on each path. Since, the values of λ covered by a path will form a closed interval, and $\mathcal{N}(a)$ is entirely contained in one path, there can not be more than one path in $\mathcal{N}$.

An edge of polyhedron $P' \times Q$ is formed by either vertex of Q and edge of P', or edge of Q and vertex of P'. Let these two type of edges be called T_1 and T_2 respectively. Clearly, λ remains constant on any edge of type T_2. However, this can not be the case for type T_1 edge in $\mathcal{N}$ as shown next.

Claim 6 *On every type T_1 edge of $\mathcal{N}$, λ strictly changes.*

Proof : From the discussion it is clear that $\mathcal{N}$ forms a path on 1-skeleton of $P' \times Q$, with alternating edges of type T_1 and T_2. Let e_1 and e_2 be two adjacent edges of type T_1 and T_2. To the contrary suppose λ is constant on e_1, and takes value a. In that case $\lambda = a$ on both e_1 and e_2 since e_2 is of type T_2. Hence $e_1 \in \mathcal{N}(a)$ and $e_2 \in \mathcal{N}(a)$ contradicting its convexity. □

Claim 6 implies that set $\mathcal{N}(a)$ consists of either a point on an edge of type T_1, or an entire edge T_2. In both the cases, there is a unique x. □

Lemma 5 implies that function f is well defined. Next we show how this function relates to the Nash equilibria of game (A, a, b).

Lemma 7 *(x, y) is a NE of game (A, a, b) iff $a = x^T \cdot c$ is a fixed-point of function f.*

Proof : Due to Lemma 3, (x, y) is a NE of game (A, a, b) if and only if $(x, y, \pi_1, \pi_2, a) \in \mathcal{N}$, where $c = x^T \cdot a$, $\pi_1 = x^T Ay$ and $\pi_2 = x^T(-A + a \cdot b^T)y$. This in turn is possible if and only if (x, y, π_1, π_2) is a solution of $LP(c)$ (Lemma 4). The later case happens only at fixed points of f. □

Thus, computing a NE of game (A, a, b) reduces to finding a fixed point of function f with one-dimensional domain. The latter is known to be doable in polynomial time using binary search, as follows:

1. Initialize: $c_1 = \min_i a_i$ and $c_2 = \max_i a_i$.
2. For $i = 1, 2$, if $f(c_i) = c_i$ then output $LP(c_i)$ and exit.
3. $c \leftarrow {}^{c_1+c_2}/2$. If $f(c) = c$ then output $LP(c)$ and exit.
4. else if $f(c) > c$ then $c_1 \leftarrow c$ else $c_2 \leftarrow c$. Go to Step 3.

Let L be the total bit length of input (A, a, b).

Theorem 8 *The algorithm terminates in at most poly(m, n, L) many iterations.*

Proof : Since, Nash equilibrium strategy profile of a bimatrix game consists of only rational numbers of polynomial size [14], fixed points of function f are also rational numbers of polynomial size (Lemma 7). The number of pivots binary search takes is at most the size of fixed points. □

3.1 Degeneracy

If polyhedron P' or Q is degenerate, then Lemma 5 does not apply, and f becomes a correspondence. However the set $f(a)$ is always convex and compact. Therefore, we can use Kakutani fixed-point theorem [8] instead and Lemma 7 will still hold. An obvious modification in the algorithm are: *i*) instead of checking $f(c) = c$, check if $c \in f(c)$ which can be done in polynomial time. *ii*) Similarly, $f(c) > c$ will imply checking if $d > c, \ \forall d \in f(c)$.

4 Discussion

The next step is to consider rank-2 games, where $A + B$ can be written as $a \cdot b^T + e \cdot f^T$. The analysis of Section 2 can be extended to get an LP with two parameters, in turn the fixed-point formulation of Section 3 will be on two-dimensional domain. However, unlike 1-D fixed-point, 2-D fixed-point computation is PPAD-hard in general [3]. It would be very interesting to know if this hardness carries back to rank-2 games, or not.

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